

# Relations between three formalisms for irrotational binary neutron stars in general relativity

Ericourgoulhon

Département d'Astrophysique Relativiste et de Cosmologie  
UPR 176 du C.N.R.S., Observatoire de Paris,  
F-92195 Meudon Cedex, France  
(22 April 1998)

Various formalisms proposed recently for irrotational binary systems in general relativity are compared and explicit relations between them are exhibited. It is notably shown that the formalisms of (i) Teukolsky, (ii) Shibata and (iii) Bonazzola et al. (as corrected by Asada) are equivalent, i.e. yield exactly the same solution, although the former two are simpler than the latter one.

PACS number(s): 95.30.Sf, 04.40.Dg, 97.60.Jd, 47.15.Hg, 04.30.Db, 04.25.Dm

## I. INTRODUCTION

Inspiral neutron star binaries are expected to be one of the strongest source of gravitational radiation for the interferometric detectors GEO600, LIGO, TAMA and VIRGO currently under construction . These systems are therefore subject to numerous theoretical studies. Among them are fully relativistic hydrodynamical treatments, pioneered by the works of Wilson et al. [1,2] and Oohara and Nakamura [3]. The most recent numerical calculations, those of Baumgarte et al. [4,5] and Marronetti et al. [6], rely on the approximations of (i) quasiequilibrium state and (ii) synchronized binaries. Whereas the first approximation is well justified up to the innermost stable orbit, the second one does not correspond to physical situations, since the gravitational-radiation driven evolution is too rapid for the viscous forces to synchronize the spin of each neutron star with the orbital motion [7,8] — as they do for ordinary stellar binaries. Rather, the fluid velocity circulation (with respect to some inertial frame) is conserved in these systems. Provided that the initial spins are not in the millisecond regime, this means that close configurations are well approximated by *irrotational* (i.e. with zero vorticity) states.

The first relativistic formulation for quasiequilibrium irrotational binaries has been given by Bonazzola, Gourgoulhon and Marck [9] (hereafter BGM). Their method is based on one aspect of irrotational motion, namely the *counter-rotation* (as measured in the co-orbiting frame) of the fluid with respect to the orbital motion. Then Teukolsky [10] and Shibata [11] gave two other formulations based on the very definition of irrotationality, which implies that the specific enthalpy times the fluid 4-velocity is the gradient of some scalar field [12] (*potential flow*). The formulation of Teukolsky is fully four-dimensional, whereas that of Shibata is three-dimensional.

The aim of the present note is to clarify the relations between these three formalisms and in particular to answer to the question: do these rather different formalisms lead to the same solution ? Of course, at the Newtonian limit, they do. However it is not obvious that this still holds in the relativistic regime. For instance, the first integral of motion — the fundamental equation which is used to compute the matter distribution in numerical procedures — is written by BGM as (Eq. (66) of Ref. [9])

$$\ln h + \frac{1}{2} \ln(N^2 - B_i B^i) + \ln \Gamma = \text{const} , \quad (1)$$

whereas for Teukolsky it reads (Eq. (54) of Ref. [10])

$$N[h^2 + D_i \psi D^i \psi]^{1/2} + B^i D_i \psi = \text{const} =: C \quad (2)$$

and for Shibata (Eq. (2.18) of Ref. [11])

$$\frac{h}{\lambda} + S^i D_i \psi = \text{const} . \quad (3)$$

In these equations,  $h$  is the fluid specific enthalpy,  $N$  is the lapse function corresponding to the 3+1 slicing of spacetime by spacelike hypersurfaces  $\Sigma_t$ ,  $B^i$  is the shift vector of rotating coordinates<sup>1</sup>,  $\Gamma$  is the Lorentz factor between the

---

<sup>1</sup>Greek (resp. Latin) indices run from 0 to 3 (resp. 1 to 3). Unless explicitly mentioned, the notations of BGM are systematically used. In particular, the shift vector  $B^i$  is the negative of  $B^i$  defined by Teukolsky [10].

fluid observer and the co-orbiting observer,  $\psi$  is the scalar potential from which the fluid 4-velocity can be derived (irrotational condition),  $D_i$  is the covariant derivative on the hypersurface  $\Sigma_t$  (related to the full spacetime covariant derivative  $\nabla_\alpha$  by Eq. (33) of Ref. [10]), and  $\lambda$  and  $S^i$  define the decomposition of the fluid velocity between a part along the helicoidal Killing vector of the quasiequilibrium assumption [9] and a part parallel to  $\Sigma_t$ <sup>2</sup>. Needless to say, Eqs. (1)-(3) differs from each other quite substantially, at least at first glance. However, we shall see that they do represent the same equation. More generally, the answer to the question raised above is *yes*, i.e. BGM, Teukolsky and Shibata formalisms are equivalent. To establish this, we will first consider Teukolsky formulation (Sect. II) and make the connection of the two other formulations with it (Sect. III and IV).

## II. TEUKOLSKY FORMALISM

Let us briefly recall and discuss Teukolsky formalism [10]. The neutron star matter is very well represented by a perfect fluid, whose stress-energy tensor writes  $T_{\alpha\beta} = (e + p)u_\alpha u_\beta + pg_{\alpha\beta}$ ,  $e$  being the fluid proper energy density,  $p$  the fluid pressure,  $u^\alpha$  the fluid 4-velocity and  $g_{\alpha\beta}$  the spacetime metric. The zero temperature approximation is fully justified for neutron star matter. In this case the fundamental energy-momentum conservation relation  $\nabla_\mu T^{\mu\alpha} = 0$  can be shown to be equivalent to the two equations

$$u^\mu \nabla_\mu (h u^\alpha) + \nabla^\alpha h = 0 , \quad (4)$$

$$\nabla_\mu (n u^\mu) = 0 , \quad (5)$$

where  $n$  is the baryon numerical density and  $h$  is the specific enthalpy:  $h = (e + p)/(m_b n)$ ,  $m_b$  being the baryon mass. Note that thanks to the First Law of Thermodynamics at zero temperature,  $h$  is also the baryon chemical potential (divided by  $m_b$ ), so that the Gibbs-Duhem relation writes  $\nabla_\alpha p = m_b n \nabla_\alpha h$ , relation which has been used to derive Eq. (4).

The *vorticity*, or *rotation 2-form*, of the fluid is defined as [13]

$$\Omega_{\alpha\beta} = P_\alpha{}^\mu P_\beta{}^\nu \nabla_{[\nu} u_{\mu]} , \quad (6)$$

where  $P_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$  is the projection tensor on the fluid rest-frame hyperplanes. Note that the definition (6) differs from that of Teukolsky (Eq. (19) in Ref. [10]), which includes a factor  $h$  in front of  $u_\alpha$ . However, it is easy to see that  $\Omega_{\alpha\beta}^{\text{Teukolsky}} = 2h \Omega_{\alpha\beta}$ , so that an *irrotational* flow can be defined either by  $\Omega_{\alpha\beta} = 0$  or by  $\Omega_{\alpha\beta}^{\text{Teukolsky}} = 0$ . By means of Eq. (4), the vorticity can be re-written as

$$\Omega_{\alpha\beta} = \frac{1}{h} \nabla_{[\beta} (h u_{\alpha]} , \quad (7)$$

so that the irrotational condition simply means that the 1-form  $h u_\alpha$  is closed. Hence there exists (at least locally) a scalar field  $\psi$  such that

$$u^\alpha = \frac{1}{h} \nabla^\alpha \psi , \quad (8)$$

i.e. the flow is *potential* [12]. The normalization relation  $u_\mu u^\mu = -1$  implies then

$$h^2 = -\nabla_\mu \psi \nabla^\mu \psi . \quad (9)$$

Thanks to this latter equation, the momentum conservation equation (4) is automatically satisfied, so that the problem reduces to finding a solution of Eq. (5), which becomes

$$\nabla_\mu \nabla^\mu \psi + \nabla^\mu \psi \nabla_\mu \ln(n/h) = 0 . \quad (10)$$

The quasiequilibrium assumption is implemented by requiring the existence of a *helicoidal Killing vector*  $l^\alpha$  [9]. Teukolsky has then shown that Eq. (9) is equivalent to Eq. (2) and that Eq. (10) becomes (Eq. (50) of Ref. [10])

---

<sup>2</sup>  $\lambda$  and  $S^i$  are respectively denoted  $u^0$  and  $V^i$  by Shibata [11].

$$D_i D^i \psi + D^i \psi D_i \ln \left( \frac{Nn}{h} \right) + \frac{1}{N^2} (C - B^i D_i \psi) \left( B^i D_i \ln \left( \frac{n}{Nh} \right) - NK \right) - \frac{1}{N^2} B^i D_i (B^j D_j \psi) = 0 , \quad (11)$$

where  $K$  is the trace of the extrinsic curvature tensor  $K_{ij}$  of the  $\Sigma_t$  hypersurfaces and  $C$  is the same constant as in Eq. (2).  $C$  is in fact defined by Eq. (39) of Ref. [10]:  $C = -l^\mu \nabla_\mu \psi$ . The three-dimensional equation (11) has to be solved in  $\psi$  to get a solution to the problem.

### III. LINKS BETWEEN SHIBATA AND TEUKOLSKY FORMALISMS

The starting point of Shibata formulation [11] is the following decomposition of the fluid 4-velocity (see footnote 2):

$$u^\alpha = \lambda(l^\alpha + S^\alpha) \quad \text{with} \quad n_\mu S^\mu = 0 , \quad (12)$$

where  $n^\alpha$  is the unit future-directed normal vector to  $\Sigma_t$ . This relation is inserted in the momentum conservation equation (4) and the result projected onto  $\Sigma_t$  to get<sup>3</sup> (Eq. (2.13) of Ref. [11] with the helicoidal symmetry taken into account)

$$D_i \left( \frac{h}{\lambda} + p_i S^i \right) + S^j (D_j p_i - D_i p_j) = 0 , \quad (13)$$

where

$$p^\alpha := h^\alpha{}_\mu (h u^\mu) , \quad (14)$$

$h_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta$  being the projection tensor onto  $\Sigma_t$  ( $p^i$  is denoted  ${}^{(3)}\tilde{u}^i$  by Shibata). Shibata defines then the irrotational condition by requiring

$$p_i = D_i \phi , \quad (15)$$

where  $\phi$  is a scalar field. Equation (13) reduces then to the first integral (3). By a somewhat lengthy calculation, Shibata shows that (15) is equivalent to  $\Omega_{\alpha\beta} = 0$ . As discussed in Sect. II, this means that the definitions of irrotationality by Shibata and Teukolsky coincide. It must then be possible to exhibit a correspondence between the two formulations.

First, inserting Eq. (8) into Eq. (14) and using Eq. (15) leads immediately to  $\phi = \psi$ . Then, the expression  $l^\alpha = N n^\alpha - B^\alpha$  (Eq. (7) of Ref. [9]) combined with Eq. (12) leads to

$$S^\alpha = \frac{1}{\lambda} h^\alpha{}_\mu u^\mu + B^\alpha = \frac{1}{\lambda h} D^\alpha \psi + B^\alpha . \quad (16)$$

Besides, the normalization relation  $u_\mu u^\mu = -1$  gives

$$\lambda = \frac{1}{N} \left( 1 + \frac{1}{h^2} D_i \psi D^i \psi \right)^{1/2} . \quad (17)$$

Inserting Eqs. (16) and (17) in Shibata's form of the first integral of motion [Eq. (3) above] gives Teukolsky's version of it [Eq. (2) above].

The equation for determining  $\psi$  given by Shibata (Eq. (2.22) of Ref. [11]) is also at first glance quite different from the equation given by Teukolsky (Eq. (11) above), since it reads

$$D_i \left( N \frac{n}{h} D^i \psi \right) + D_i (N n \lambda B^i) = 0 . \quad (18)$$

However, Eqs. (17) and (2) yield the following expression of  $\lambda$ :

$$\lambda = \frac{1}{N^2 h} (C - B^i D_i \psi) , \quad (19)$$

which, once reported in Eq. (18), gives exactly Eq. (11).

---

<sup>3</sup>Note that Eqs. (2.10) and (2.11) of Ref. [11] are not correct: their right-hand-side must be supplemented by respectively  $-h n_\mu u^\mu K_{ij} V^j$  and  $+h n_\mu u^\mu K_{ij} V^j$  (using Shibata's notation for  $V^j$ ). Fortunately, these corrections cancel each other when summing Eqs. (2.10) and (2.11), so that Shibata's final result (Eq. (2.13) of Ref. [11]) is correct.

#### IV. LINKS BETWEEN BGM AND TEUKOLSKY FORMALISMS

The BGM formulation [9] is based on the fluid 3-velocity with respect to the co-orbiting frame. This later frame is that of the observer whose worldlines are the trajectories of the Killing vector  $l^\alpha$ , i.e. whose 4-velocity is  $v^\alpha = e^{-\Phi} l^\alpha$ , with  $\Phi = 1/2 \ln(N^2 - B_i B^i)$ . The 3-velocity  $V^\alpha$  with respect to that observer is defined by the orthogonal decomposition

$$u^\alpha = \Gamma(V^\alpha + v^\alpha) \quad \text{with} \quad v_\mu V^\mu = 0. \quad (20)$$

The momentum conservation equation (4) is then re-expressed as an equation for  $V^\alpha$ , which is a relativistic generalization of Euler equation in a rotating frame. The basic idea of BGM is to define a counter-rotating motion as a motion for which the Coriolis term is canceled by the  $(\nabla \wedge V) \wedge V$  term coming from the advection term  $V \cdot \nabla V$ , where the wedge product is defined in the co-orbiting observer rest frame:  $(X \wedge Y)^\alpha = v_\sigma \epsilon^{\sigma\alpha}_{\mu\nu} X^\mu Y^\nu$ . It has been however noted by Asada [14] that the condition given by BGM is not enough to fully determine the velocity field. In fact the correct condition is obtained by setting  $F = 0$  in Eq. (50) of BGM [9], so that this latter becomes

$$(\nabla \wedge e^{-\Phi} V)^\alpha = -2e^{-\Phi} \omega^\alpha, \quad (21)$$

which is identical to Eq. (4.12) of Asada [14]. In this equation  $\omega^\alpha$  is the rotation vector of the co-orbiting observer:  $2\omega^\alpha := v_\sigma \epsilon^{\sigma\alpha\mu\nu} \nabla_\mu v_\nu$ . If the counter-rotation condition (21) is satisfied, the Euler equation reduces then simply to the first integral (1). The baryon number conservation equation (5) gives an additional equation, for the divergence of  $V^\alpha$ :

$$\nabla_\mu V^\mu + V^\mu \nabla_\mu \ln(n\Gamma) = 0. \quad (22)$$

The BGM procedure is to perform a 3+1 decomposition of Eqs. (21)-(22) and to introduce scalar and vector potentials so that Eq. (22) gives a scalar Poisson-type equation and Eq. (21) gives a vector Poisson equation. This contrasts with Teukolsky or Shibata formalisms, which require to solve only one scalar Poisson-type equation (Eq. (11) or Eq. (18)). It has been however noted by Asada [14] that the counter-rotation condition (21) is equivalent to the irrotational condition  $\Omega_{\alpha\beta} = 0$ . Hence there must exist a re-interpretation of the BGM formulation in terms of Teukolsky's one.

In fact, if the irrotational condition (8) holds, one can see easily, by means of the relation  $\Gamma = -v_\mu u^\mu$ , that

$$\Gamma = \frac{C}{he^\Phi}, \quad (23)$$

where  $C$  is the same constant as in Eq. (2). Taking the logarithm of expression (23) gives directly the first integral (1).

Combining Eqs. (20), (8) and (23) yields

$$V^\alpha = \frac{e^\Phi}{C} \nabla^\alpha \psi - v^\alpha, \quad (24)$$

from which one can derive the following expression for the part of  $V^\alpha$  parallel to  $\Sigma_t$  (cf. Sect. V.B of BGM):

$$W^i = \frac{e^\Phi}{C} D^i \psi + e^{-\Phi} B^i. \quad (25)$$

It is immediate to verify that the 3-velocity given by Eq. (24) obeys to the counter-rotation condition (21). Note that this way of proceeding is more straightforward than the converse one: it took a full appendix in [14] to verify that counter-rotation implies irrotational flow.

#### V. CONCLUSION

We have shown explicit relations between the formalisms introduced by BGM, Teukolsky and Shibata to treat irrotational neutron star binaries. All these formalisms are equivalent in the sense that they lead to exactly the same solution. Teukolsky's formulation [10] is the simplest one because it is directly based on the four-dimensional definition of irrotational motion. Shibata's one [11] contains some complications induced by its three-dimensional character, but both formulations result in the same type of scalar Poisson-like equation governing the fluid motion (Eqs. (11) and (18) above). On the contrary, BGM formulation [9] (corrected by Asada [14]) is more complicated

since it requires the resolution of an additional vector Poisson equation. Therefore, for numerical studies, Teukolsky or Shibata procedure should be preferred.

---

- [1] J. Wilson and G. Mathews, Phys. Rev. Lett. **75**, 4161 (1995).
- [2] J. Wilson, G. Mathews, P. Marronetti, Phys. Rev. D **54**, 1317 (1996).
- [3] K. Oohara and T. Nakamura, in *Relativistic Gravitation and Gravitational Radiation*, edited by J.-A. Marck and J.-P. Lasota (Cambridge University Press, Cambridge, England, 1997).
- [4] T.W. Baumgarte, G.B. Cook, M.A. Scheel, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. Lett. **79**, 1182 (1997).
- [5] T.W. Baumgarte, G.B. Cook, M.A. Scheel, S.L. Shapiro, and S.A. Teukolsky, to be published (preprint: gr-qc/9709026).
- [6] P. Marronetti, G.J. Mathews, and J.R. Wilson, preprint gr-qc/9803093.
- [7] C.S. Kochanek, Astrophys. J. **398**, 234 (1992).
- [8] L. Bildsten and C. Cutler, Astrophys. J. **400**, 175 (1992).
- [9] S. Bonazzola, E. Gourgoulhon, and J.-A. Marck, Phys. Rev. D **56**, 7740 (1997).
- [10] S.A. Teukolsky, to appear in Astrophys. J. (preprint: gr-qc/9803082).
- [11] M. Shibata, to appear in Phys. Rev. D (preprint: gr-qc/9803085).
- [12] L.D. Landau and E.M. Lifchitz, *Mécanique des fluides*, § 134 (Editions Mir, Moscow, 1989).
- [13] J. Ehlers, Proc. Math. Nat. Sci. Sect. Mainz Academy of Science and Litterature **11**, 792 (1961); English translation in Gen. Rel. Grav. **25**, 1225 (1993).
- [14] H. Asada, to appear in Phys. Rev. D (preprint: gr-qc/9804003).